Advanced QFT @ EPFL 2024

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— Path Integral —

Let's consider first a QM system with just one degree of freedom, i.e. one pair of conjugate variables (Q, P)

$$(1)\quad \begin{cases} [Q,P]=i\,, \quad [Q,Q]=0=[P,P] \\ Q|q\rangle = q|q\rangle\,, \quad P|p\rangle = p|p\rangle\,, \quad \int dq\,|q\rangle\langle q| = \mathbb{1} = \int \frac{dp}{2\pi}|p\rangle\langle p| \\ \langle p|q\rangle = e^{-ipq} \quad \left(\langle q|q'\rangle = \delta(q-q') \qquad \langle p|p'\rangle = 2\pi\delta(p-p')\right) \end{cases}$$

with Hamiltonian

$$(2)\quad \boxed{H(P,Q) = \frac{P^2}{2m} + V(Q)}$$

which allows us the define the following τ- evolution Operator:

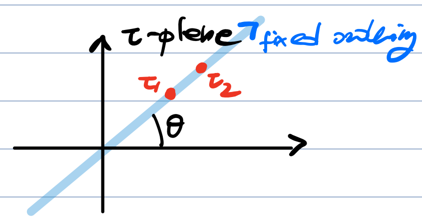
$$(3)\quad \boxed{U(\tau) \equiv \exp(-\tau H) \qquad \tau = t e^{i\theta} \qquad t\in\mathbb{R} \qquad -\frac{\pi}{2} < \theta < \pi/2 \quad \cos\theta > 0}$$

which is nothing but the time evolution operator for $\theta \to \pi/2$, i.e. $\tau \to it$

In the following we will need the notion of

"τ- ordering" = ordering in t at fixed θ

$\boxed{\tau_2 > \tau_1 \iff t_2 > t_1}$ [[since $\cos\theta > 0$, also $\tau_2 > \tau_1 \iff Re(\tau_2) > Re(\tau_1)$]]



Analogously to time evolved operators, we can define τ - evolved operators $\mathcal{O}(\tau)$ from $\mathcal{O} = \mathcal{O}(P, Q)$

$$(4)\quad \begin{cases} \mathcal{O}(\tau) \equiv U^{-1}(\tau)\,\mathcal{O}(P,Q)\,U(\tau) = \mathcal{O}(P(\tau), Q(\tau)) \\ U^{-1}(\tau) = \exp(+\tau H) \end{cases}$$

With their bra & ket eigenvectors obtained by acting with $U^{-1}(\tau)$ and $U(\tau)$ respectively, e.g.

$$(5)\quad \begin{aligned} |q,\tau\rangle &= U^{-1}(\tau)|q\rangle & \langle q,\tau| &= \langle q|\,U(\tau) \\ |p,\tau\rangle &= U^{-1}(\tau)\,|p\rangle & \langle p,\tau| &= \langle p|\,U(\tau) \end{aligned}$$

so that $Q(\tau)\,|q,\tau\rangle = U^{-1} Q\, U\, U^{-1}|q\rangle = q\,U^{-1}|q\rangle = q\,|q,\tau\rangle$ and so on:

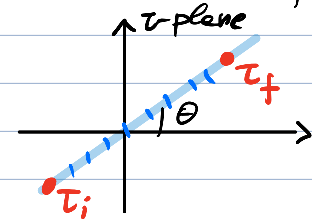
Let's consider Q-eigenstates matrix elements at ordered times:

$$(6)\quad \begin{cases} \tau_f > \tau_i \quad \text{i.e.} \quad t_f > t_i \quad \text{i.e.} \quad \boxed{\mathrm{Re}\,(\tau_f - \tau_i) > 0} \\ \langle q_f, \tau_f \,|\, q_i, \tau_i\rangle = \langle q_f|\exp\left(-(\tau_f-\tau_i)H\right)|q_i\rangle = \boxed{\langle q_f|\,U(\tau_f-\tau_i)\,|q_i\rangle} \end{cases}$$

Let's rewrite this τ-evolution from τ_i to τ_f in several small steps

$$(7)\quad U(\tau_f-\tau_i) = \left(U(\Delta\tau)\right)^N = \underbrace{U(\Delta\tau)\ldots U(\Delta\tau)}_{N\text{-times}}$$

$$\Delta\tau = \frac{\tau_f - \tau_i}{N}$$



So that inserting $(N-1)$-times complete states

$$(8)\quad \mathbb{1} = \int dq\,|q\rangle\langle q| = \int dq\;|q,\tau\rangle\langle q,\tau|$$

we can express everything in terms of a small τ-evolution, repeated several times

$$(9)\quad \langle q_f|\,U(\tau_f-\tau_i)\,|q_i\rangle = \int dq_{N-1}\ldots dq_1\,\langle q_f|\,U(\Delta\tau)\,|q_{N-1}\rangle\ldots\langle q_1|\,U(\Delta\tau)\,|q_i\rangle$$

$$= \int dq_{N-1}\ldots dq_1\;\langle q_f,\tau_f\,|\,q_{N-1},\underbrace{\tau_f-\Delta\tau}_{\tau_{N-1}}\rangle\ldots\langle q_2,\underbrace{\tau_i+2\Delta\tau}_{\tau_2}\,|\,q_1,\underbrace{\tau_i+\Delta\tau}_{\tau_1}\rangle\langle q_1,\underbrace{\tau_i+\Delta\tau}_{\tau_1}\,|\,q_i\,\tau_i\rangle$$

so that one needs only the infinitesimal transfer matrix: $\langle q', \tau+\Delta\tau | q, \tau\rangle$

$$\text{(10)}\quad \langle \overbrace{q_{n+1}, \underbrace{\tau_n+\Delta\tau}_{\tau_{n+1}} | q_n}^{\tau\text{-ordered}}, \tau_n\rangle \underset{\text{def}}{=} \langle q_{n+1}| \exp\left[-\left(\frac{P^2}{2m}+V(Q)\right)\Delta\tau\right]|q_n\rangle$$

$$\underset{e^{A+B}=e^A e^B\left(1-\frac{1}{2}[A,B]+\dots\right)}{=} \langle q_{n+1}| e^{-\Delta\tau\frac{P^2}{2m}}|q_n\rangle\, e^{-V(q_n)\Delta\tau}\left(1+O(\Delta\tau^2)\right)$$

$$\underset{\mathbb{1}=\int\frac{dp}{2\pi}|p\rangle\langle p|}{=} \int\frac{dp_n}{2\pi}\langle q_{n+1}|p_n\rangle\langle p_n|q_n\rangle\, e^{-\Delta\tau\left(\frac{p_n^2}{2m}+V(q_n)\right)}\left(1+O(\Delta\tau^2)\right)$$

$$\underset{\langle p|q\rangle=e^{-ipq}}{=} \int\frac{dp_n}{2\pi}\, e^{ip_n(q_{n+1}-q_n)-\Delta\tau\left(\frac{p_n^2}{2m}+V(q_n)\right)}\left(1+O(\Delta\tau^2)\right)$$

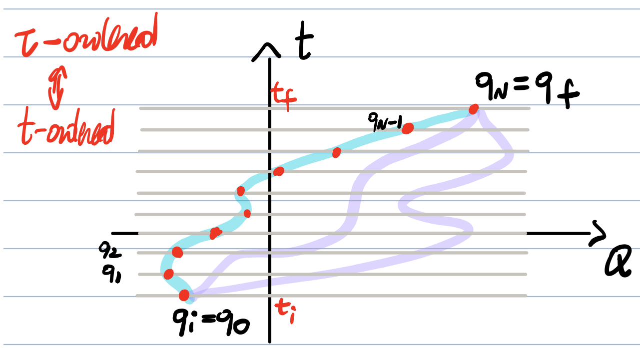
$$= \int\frac{dp_n}{2\pi}\, e^{-\Delta\tau\left[\frac{p_n^2}{2m}+V(q_n)-ip_n\frac{(q_{n+1}-q_n)}{\Delta\tau}\right]}\left(1+O(\Delta\tau^2)\right)$$

So that neglecting $O(\Delta\tau^2)$-error for $\Delta\tau\to 0$

$$\text{(11)}\quad \boxed{\langle q_{n+1}, \tau_n+\Delta\tau | q_n, \tau_n\rangle = \int\frac{dp_n}{2\pi}\, e^{-\Delta\tau\left[\frac{p_n^2}{2m}+V(q_n)-ip_n\frac{(q_{n+1}-q_n)}{\Delta\tau}\right]}}$$

Which inserted in (9), given by τ-ordered integrals, gives

$$\text{(12)}\quad \langle q_f, \tau_f | q_i\, \tau_i\rangle = \int \frac{dp_0}{2\pi}\, dq_1 \frac{dp_1}{2\pi}\, dq_2 \frac{dp_2}{2\pi}\cdots dq_{N-1}\frac{dp_{N-1}}{2\pi}\, e^{-\Delta\tau\left[\sum_{n=0}^{N-1}\frac{p_n^2}{2m}+V(q_n)-ip_n\frac{(q_{n+1}-q_n)}{\Delta\tau}\right]}$$



The continuum limit: $\Delta\tau\to 0$ $N\to\infty$

with $p(\tau)$ & $q(\tau)$ interpolating functions

at p_n & q_n for $\tau = \tau_i + n\Delta\tau$

continuum limit

$$(13)\quad \langle q_f,\tau_f|q_i,\tau_i\rangle \longrightarrow \int_{\substack{q(\tau_i)=q_i=q_0\\ q(\tau_f)=q_f=q_N}}[dp(\tau)][dq(\tau)]\, e^{-\int_{\tau_i}^{\tau_f}\Big(\underbrace{\frac{p^2(\tau)}{2m}+V(q(\tau))}_{H(p,q)}-i\,p(\tau)\,\dot q(\tau)\Big)d\tau}$$

$$= \int_{\substack{q(\tau_i)=q_i=q_0\\ q(\tau_f)=q_f=q_N}}[dp(\tau)][dq(\tau)]\,\exp\Big\{-\int_{\tau_i}^{\tau_f}d\tau\big(H(p(\tau),q(\tau))-i\,p(\tau)\,\dot q(\tau)\big)\Big\}$$

fixed Boundary conditions

Amplitude from q_i at τ_i to q_f at τ_f given by an integral over all paths in phase space (p,q) with weight $\exp\int-(H-ip\dot q)d\tau$

Remark: everything has been done for arbitrary $-\pi/2<\theta<\pi/2$: $\tau=te^{i\theta}$

- $\theta\longrightarrow 0$ "Euclidean Path integral"
- $\theta\rightarrow \pi/2$ "Minkowski Path integral"

We can simplify things by recalling that each p_n enters quadratically in the exponential in (11) & (12), providing a single gaussian integral

$$(14)\quad \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\,e^{-\frac12Ap^2+Bp} = \int_{-\infty}^{+\infty}\frac{dp}{2\pi}\,e^{-\frac12A(p-\frac{B}{A})^2+\frac{B^2}{2A}} = e^{\frac{B^2}{2A}}\int_{-\infty}^{+\infty}\frac{d\bar p}{2\pi}\,e^{-\frac12A\bar p^2} \underset{\mathrm{Re}\,A>0}{=} e^{\frac{B^2}{2A}}\cdot\frac{1}{\sqrt{2\pi A}}$$

$$(15)\quad \boxed{\int_{-\infty}^{+\infty}\frac{dp}{2\pi}\,e^{-\frac12Ap^2+Bp} \underset{\mathrm{Re}\,A>0}{=} e^{B^2/2A}\,\frac{1}{\sqrt{2\pi A}}}$$

gaussian integration

In our case in (11) $\boxed{A = \frac{\Delta\tau}{m} \quad B = i(q_{n+1} - q_n)}$ $\Rightarrow$ Re $A = \frac{\cos\theta\, \Delta t}{m} > 0$

time ordered & $\pi/2 < \theta < \pi/2$

$$(16) \quad \int_{-\infty}^{+\infty} \frac{dp_n}{2\pi} e^{-\Delta\tau\left(\frac{p_n^2}{2m} - i p_n \left(\frac{q_{n+1}-q_n}{\Delta\tau}\right)\right)} = e^{-\frac{m}{2}\left(\frac{q_{n+1}-q_n}{\Delta\tau}\right)^2 \Delta\tau} \cdot \sqrt{\frac{m}{2\pi\Delta\tau}}$$

In (12) there are N such Gaussian integrals

$$(17) \quad \langle q_f, \tau_f | q_i \tau_i \rangle = \int \frac{dp_0}{2\pi} dq_1 \frac{dp_1}{2\pi} dq_2 \frac{dp_2}{2\pi} \cdots dq_{N-1} \frac{dp_{N-1}}{2\pi} e^{-\Delta\tau\left[\sum_{n=0}^{N-1} \frac{p_n^2}{2m} + V(q_n) - i p_n \frac{(q_{n+1}-q_n)}{\Delta\tau}\right]}$$

$$= \int dq_1 \ldots dq_{N-1} \left(\sqrt{\frac{m}{2\pi\Delta\tau}}\right)^N \exp\left[-\Delta\tau \sum_{n=0}^{N-1} \frac{m}{2}\left(\frac{q_{n+1}-q_n}{\Delta\tau}\right)^2 + V(q_n)\right]$$

$$(17) \quad \boxed{\langle q_f, \tau_f | q_i \tau_i \rangle \xrightarrow[\text{contin. limit}]{} = \int [dq(\tau)] \exp\left\{-\int_{\tau_i}^{\tau_f} d\tau \left(\frac{m}{2}\dot{q}^2(\tau) + V(q_n)\right)\right\} \quad \begin{cases} q_0 = q_i = q(\tau_i) \\ q_N = q_f = q(\tau_f) \end{cases} \leftarrow \text{fixed boundary conditions}}$$

Comment: The overall normalization, even if divergent, will not matter in the following, see below.

Remark:

$\tau = t e^{i\theta} \xrightarrow[\theta \to 0]{} t$ "Euclidean" $\quad \exp\left\{-\int_{t_i}^{t_f} dt \left(\frac{m}{2}\dot{q}^2(t) + V(q(t))\right)\right\}$ Energy > 0

$\tau = t e^{i\theta} \xrightarrow[\theta \to \pi/2]{} it$ "Minkowski" $\quad \exp\left\{+i\int_{t_i}^{t_f} dt \underbrace{\left(\frac{m}{2}\dot{q}^2(t) - V(q)\right)}_{\text{Lagrangian}}\right\}$

— Following the $i\varepsilon$ — $\quad \theta \to \pi/2^-$

$$\tau = t e^{i\theta} = t e^{i(\pi/2 - \varepsilon)} \to i t (1 - i\varepsilon) \qquad \varepsilon > 0$$

(18) $$\langle q_f, \tau_f | q_i \tau_i \rangle \xrightarrow[\theta \to \pi/2^-]{\text{Mink.}} \int_{\substack{q_i = q(\tau_i) \\ q_f = q(\tau_f)}} [dq(t)] \exp\left\{ i \int_{t_i}^{t_f} dt \, (1-i\varepsilon) \left(m \frac{\dot{q}^2}{2(1-i\varepsilon)^2} - V(q) \right) \right\}$$

$$= \int_{\substack{q_i = q(\tau_i) \\ q_f = q(\tau_f)}} [dq(t)] \exp\left\{ i \underbrace{\int_{t_i}^{t_f} dt \left(m \frac{\dot{q}^2}{2} - V(q) \right)}_{S_{\text{Mink}}} + i\varepsilon \underbrace{\left(m \frac{\dot{q}^2}{2} + V(q) \right)}_{\text{Energy} > 0} \right\}$$

Positive convergent factor

Remark: Both for Euclidean & Mink. is positivity of Energy that ensures convergence factor

"S_{Mink} always has small imaginary part"

(19) $$\boxed{S_{\text{Euclid}}(t) \xrightarrow[t \to it]{} S_{\text{Mink}}(t) + i\varepsilon \qquad \varepsilon > 0}$$

Notation: In the following we call $S(\tau_i, \tau_f) \equiv \int_{\tau_i}^{\tau_f} d\tau \left(m \frac{\dot{q}^2}{2} + V(q) \right)$ with $\tau = t e^{i\theta}$ generic, and just $S[q(\tau)]$ the $\tau_i \to -\infty$ $\tau_f \to +\infty$

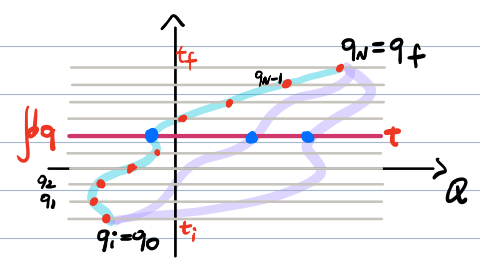
Question: What about expectation values? $\langle q_f \tau_f | \mathcal{O}(P(\tau), Q(\tau)) | q_i \tau_i \rangle$

Answer: just split the τ-evolution: from $\tau_i \to \tau$ first, then $\tau \to \tau_f$

Example: $\mathcal{O} = Q(\tau)$ with $\tau_f < \tau < \tau_i$ $\quad (t_i < t < t_f)$

(20) $$\langle q_f, \tau_f | Q(\tau) | q_i, \tau_i \rangle = \int dq \, \langle q_f \tau_f | q, \tau \rangle \langle q, \tau | q_i, \tau_i \rangle \, q$$

$$\langle q_f | e^{(\tau-\tau_i)H} Q \, e^{-(\tau-\tau_i)H} | q_i \rangle \qquad \mathbb{1} = \int dq \, |q\rangle\langle q|$$



$$= \int dq \int_{\substack{q(\tau) = q \\ q(\tau_f) = q_f}} [dq] \int_{\substack{\bar{q}(\tau) = q \\ \bar{q}(\tau_i) = q_i}} [d\bar{q}] \; q \; \underbrace{e^{-S(\tau_f, \tau)} \, e^{-S(\tau, \tau_i)}}_{e^{-S(\tau_f, \tau_i)}}$$

(21) $$\langle q_f\,\tau_f|\,Q(\tau)\,|q_i\,\tau_i\rangle = \int\limits_{\substack{q(\tau_i)=q_i\\ q(\tau_f)=q_f}}[dq]\,\exp\{-S[\tau_f,\tau_i]\}\,q(\tau)$$

Obviously, with more Q insertions it's just the same, up to more splitting & integrations — as long as they are time ordered —

(22) $$\boxed{\langle q_f,\tau_f|\,T\,(Q(\tau_1)\ldots Q(\tau_n))\,|q_i,\tau_i\rangle = \int\limits_{\substack{q(\tau_i)=q_i\\ q(\tau_f)=q_f}}[dq(t)]\,q(\tau_1)\ldots q(\tau_n)\,e^{-S(\tau_i,\tau_f)}}$$

— Vacuum Projection —

The dependence on the boundary data q_f and q_i can be removed by taking the large time limits $\tau_f \to +\infty$ $\tau_i \to -\infty$

(23) $$\lim_{\substack{\tau_f\to+\infty\\ \tau_i\to-\infty}} \langle q_f,\tau_f|\,T\,Q(\tau_1)\ldots Q(\tau_n)\,|q_i,\tau_i\rangle$$

$$= \lim_{\substack{\tau_f\to+\infty\\ \tau_i\to-\infty}} \sum_{\ell\, m} \langle q_f|e^{-\tau_f H}|E_\ell\rangle\langle E_\ell|\,T\,Q(\tau_1)\ldots Q(\tau_n)\,|E_m\rangle\langle E_m|e^{+\tau_i H}|q_i\rangle$$

$$e^{-(\tau_f-\tau_i)E_0}\left(1+O(e^{-\underbrace{(E_1-E_0)}_{>0}\underbrace{(\tau_f-\tau_i)}_{\mathrm{Re}(\tau_f-\tau_i)\to+\infty}})\right)$$

$\to 0$ exponentially

E_1: first state above E_0

$$\Downarrow$$

(24) $$= \underbrace{\langle q_f|E_0\rangle\langle E_0|q_i\rangle}_{\langle q_f;\infty|q_i,-\infty\rangle}\, e^{-(\tau_f-\tau_i)E_0}\,\langle E_0|\,T\,Q(\tau_1)\ldots Q(\tau_n)\,|E_0\rangle$$

Remark: The large-time limit always projects on ground state $|E_0\rangle$ of minimal energy as long as $E_i-E_0>0$ (strictly) and $|q_{i,f}\rangle$ are not orthogonal to $|E_0\rangle$. To make this argument fully rigorous one usually needs first to work at finite volume Vol so that E-spectrum is discrete and $E_i-E_0>0$ indeed, then take $\tau_f-\tau_i\to\infty$ first followed by Vol $\to+\infty$ at the end.

$$(25)\quad \frac{\langle q_f,+\infty|TQ(\tau_1)\ldots Q(\tau_n)|q_i,-\infty\rangle}{\langle q_f,+\infty|q_i,-\infty\rangle} = \int_{\substack{q(-\infty)=q_i\\ q(+\infty)=q_f}}[dq(t)]\, e^{-S[q(t)]}\, q(\tau_1)\ldots q(\tau_n) \Big/ \int_{\substack{q(-\infty)=q_i\\ q(+\infty)=q_f}}[dq]\, e^{-S[q(\tau)]}$$

$|E_0\rangle \equiv |0\rangle$

$$= \langle 0|TQ(\tau_1)\ldots Q(\tau_n)|0\rangle$$

assumed $\langle 0|0\rangle = 1$ in (23)-(24)
if not just divide by $\langle 0|0\rangle$

Comments:

- We assumed $\langle 0|0\rangle = 1$ in (23)-(24). If want to change normalization divide by $\langle 0|0\rangle$. Notice, moreover that $|0\rangle$ is true ground state system, we did nowhere any perturbative expansion, we just assumed the system was gapped.
- One can even calculate the ground state energy.

$$(26)\quad E_0 = \lim_{\tau_f-\tau_i\to+\infty} -\frac{1}{(\tau_f-\tau_i)}\log\langle q_f,\tau_f|q_i,\tau_i\rangle = \lim_{\tau_f-\tau_i\to\infty} -\frac{1}{(\tau_f-\tau_i)}\log\left(\int[dq]\,e^{-S}\right)$$

$$\left(\int dq\, e^{-S} \to e^{-E_0(\tau_f-\tau_i)/\hbar}\right)$$

Moreover, since r.h.s. (25) neither depends on $q_i = q(-\infty)$ nor on $q_f = q(+\infty)$, we can integrate over them to have a nicer looking expression without boundary data: $\int dq_f \int dq_i \int_{\substack{q(-\infty)=q_i\\ q(+\infty)=q_f}}[dq] = \int [dq]$ with no restriction

$$(27)\quad \frac{\int[dq]\, q(\tau_1)\ldots q(\tau_n)\, e^{-S[q]}}{\int[dq]\, e^{-S[q]}} = \langle 0|TQ(\tau_1)\ldots Q(\tau_n)|0\rangle$$

$\leftarrow$ no boundary restriction

(zero temp. limit of stat. mech.)

Comment: if we set instead another (equivalent, since it does not depend on it) $\left(\int dq\,\langle q,+\infty|TQ\ldots Q|q,-\infty\rangle\right)$ with periodic boundary conditions $q_i = q_f = q$ at $\tau_i = -\tau/2 = -\tau_f$ and int. over q, we get the statist. mechanics expression as $\theta\to 0$ $\lim_{\tau\to\infty} \mathrm{Tr}\left(TQ(t_1)\ldots Q(t_n)\, e^{-tH}\right)/\mathrm{Tr}\, e^{-tH}) = \frac{\int dq\, q(t_1)\ldots e^{-S_{euc}}}{\int dq\, e^{-S_{euc}}}$

Let's look at one important paradigmatic example:

— 1D Harmonic Oscillator via Path Integral: Euclid → Mink. —

Given $H = \frac{P^2}{2m} + \frac{\Omega^2}{2} Q^2 \xrightarrow[\theta\to 0]{} S_{Euc} \underset{\tau=t}{=} \int d\tau \frac{1}{2} q(\tau) \underbrace{\left[-\frac{d^2}{d\tau^2} + \Omega^2\right]}_{\parallel \; D_{Euc}} q(\tau)$

(28) $D_{Euc} = -\frac{d^2}{d\tau^2} + \Omega^2$ Linear diff. op with positive eigenvalues $\hookrightarrow D_{Euc} e^{i\omega t} = (\omega^2 + \Omega^2) e^{i\omega t}$

what is the 2pt-function $\langle 0 | T Q(\tau_1) Q(\tau_2) | 0 \rangle = ?$

To this end, let's first calculate the functional generator $Z[J]$ in Euclidean settings known as "partition function" (functional generator all diagrams in Mink)

$$(29) \begin{cases} Z[J] = \int [dq] \, e^{-S_{Euc} + \int J(\tau) q(\tau) d\tau} \\ \frac{1}{Z[J]} \frac{\delta Z[J]}{\delta J(\tau_1) \delta J(\tau_2)}\Big|_{J=0} = \langle 0 | T Q(\tau_1) Q(\tau_2) | 0 \rangle \end{cases}$$

There are various ways to perform the calculations of $Z[J]$ here:

(A) Gaussian Integral again:

the (29) is a Gaussian Integral of the type

$$(30) \quad \boxed{\int \left(\prod_n dx_n\right) \exp\left(-\frac{1}{2} x_i A_{ij} x_j + x_i J_i\right) \propto \exp\left(\frac{1}{2} J_i A^{-1}_{ij} J_j\right) \frac{1}{\sqrt{\det A}}}$$

[This comes about just from Eq.(15) for single Gaussian int. $\int_{-\infty}^{+\infty} dx \, e^{-\frac{1}{2}Ax^2 + Bx} = e^{\frac{B^2}{2A}} \sqrt{\frac{2\pi}{A}}$

where one first diagonalizes $A_{ij} = (R^T A_{Diag} R)_{ij}$ by changing int. variables $Rx = y$

with orthogonal R, $R^T R = \mathbb{1} \Rightarrow (30) = \int \prod_n dy_n \exp\left(-\frac{1}{2} y_i^2 A_i^{Diag} - y_i R_{ij} J^j\right) \xrightarrow[A_i = A_i^{Diag}]{B_i = -R_{ij} J_j} \exp\Big(\underbrace{\frac{1}{2} R_{ij} J_j A^{-1}_{Diag\,i} R_{ik} J_k}_{J \cdot A^{-1} \cdot J}\Big) \underbrace{\sqrt{\prod_n \frac{2\pi}{A_n^{diag.}}}}_{\det A}$]

where $D_{euc}(\tau) = \delta(\tau'-\tau)\,(-\partial_\tau^2+\Omega^2)$ is the A-operator

(31) $$Z[J] \propto \exp\left\{ \frac{1}{2}\int J(\tau')\, D^{-1}_{euc}(\tau'-\tau)\, J(\tau)\right\} \frac{1}{\sqrt{\det D}}$$

↑ Greens function $G_{euc}(\tau'-\tau)$

(32) $$\int D(\tau-\tau')\, G_{euc}(\tau''-\tau')\, d\tau' = \delta(\tau-\tau')$$

$\Downarrow$

(33) $$(\omega^2+\Omega^2)\,\hat{G}_{euc}(\omega) = 1 \quad\Rightarrow\quad \boxed{G_{euc}(\tau'-\tau) = \int \frac{d\omega}{2\pi}\, \frac{e^{-i\omega(\tau'-\tau)}}{\omega^2+\Omega^2}}$$

(34) $$Z[J] \propto \exp\left\{ \frac{1}{2}\int \hat{J}^*(\omega)\, \frac{1}{\omega^2+\Omega^2}\, J(\omega)\, \frac{d\omega}{2\pi}\right\} \frac{1}{\sqrt{\det D}}$$

$\Downarrow$

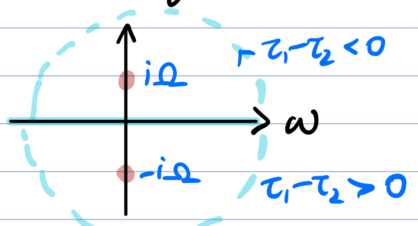
(35) $$\boxed{\langle 0|\,T\,Q(\tau_1)\,Q(\tau_2)\,|0\rangle = \frac{1}{Z[J]}\, \frac{\delta^2 Z}{\delta J(\tau_1)\,\delta J(\tau_2)}\Big|_{J=0} = D^{-1}_{euc}(\tau_1-\tau_2) = \int \frac{d\omega}{2\pi}\, \frac{e^{-i\omega(\tau_1-\tau_2)}}{\omega^2+\Omega^2}}$$

2pt-function QM 1D Harmonic Oscillator

Comment: In the discretized form, before taking the continuum limit we would have in (31) $\int \left(\prod_k dq_k\right) \exp\left(-\sum_{ij} \frac{\Delta\tau}{2}\, q_i\,(-\Delta_{ij}+\Omega^2\delta_{ij})\,q_j\right)$ $\quad \Delta_{ij} = \partial^2_{ij}$

where ∂_{ij} is the lattice derivative $\partial_{ij} = \dfrac{\delta_{j+1,i}-\delta_{j,i-1}}{2\Delta\tau}$

Incidentally: The integration in (35) can be done explicitly via residue th.



$$\theta(\tau_1-\tau_2)\,(-2\pi i) \lim_{\omega\to -i\Omega} \frac{\cancel{(\omega+i\Omega)}\, e^{-i\omega(\tau_1-\tau_2)}}{2\pi\,\cancel{(\omega+i\Omega)}\,(\omega-i\Omega)} + \theta(\tau_2-\tau_1) \lim_{\omega\to i\Omega} \frac{2\pi i\,\cancel{(\omega-i\Omega)}\, e^{-i\omega(\tau_1-\tau_2)}}{2\pi\,(\omega+i\Omega)\,\cancel{(\omega-i\Omega)}}$$

1D Harmonic Oscill. 2pt funct.

(36) $$\boxed{\langle 0|\,T\,Q(\tau_1)\,Q(\tau_2)\,|0\rangle = \frac{e^{-\Omega|\tau_1-\tau_2|}}{2\Omega} \equiv \bar{G}_{euc}(|\tau_1-\tau_2|)}$$

(← function of $\tau_1-\tau_2$ not analytic at $\tau_1=\tau_2$)

← exp suppression $|\tau_1-\tau_2| \gg \Omega$

(B) Solving Free Eq. of motion

Write the most general trajectory $q(\tau)$ as $\underset{\text{classical motion}}{q(\tau)}$ + extra, denoted $\pi(\tau)$

(37a) $q(\tau) = q_{cl}(\tau) + \pi(\tau)$

(37b) $-\frac{\delta S_{euc}}{\delta q(\tau)} + J(\tau)\Big|_{q=q_{cl}} = 0$ That is, $q_{cl}(\tau)$ makes linear variation vanishing

(37b) $\Rightarrow$ $\mathbb{D}_{euc}\cdot q(\tau)\big|_{q=q_{cl}} = J(\tau)$ $\Rightarrow$ $q_{cl} = \mathbb{D}_{euc}^{-1}\cdot J = \int G_{euc}(\tau-\tau')J(\tau')\,d\tau'$

Plugging (37a) into the path integral and using the linear terms in π vanish by construction

(38) $S_{eucl} - \int J\,q\,d\tau = \int d\tau\, \frac{1}{2}(q_{cl}+\pi)\overset{\text{“}G_{euc}^{-1}\text{”}}{\mathbb{D}_{euc}}(q_{cl}+\pi) - J(\tau)(q_{cl}+\pi)\Big|_{q_{cl}=\mathbb{D}_{euc}^{-1}\cdot J}$

$\propto \exp\left(\frac{1}{2}\int J\cdot G_{euc}\cdot J\,d\tau\right)$ with proportionality factor J-independent

$\Downarrow$

(39) $\langle 0|T\,Q(\tau_1)\,Q(\tau_2)|0\rangle = G_{euc}(\tau_1-\tau_2) = \mathbb{D}_{euc}^{-1}$ ok

— Continuation to Minkowski —

How do we recover Mink. 2pt-function? We need to send $t \to it$ keeping the ordering fixed (a rigid rotation of θ from $0 \to \pi/2^-$), and

since $G_{eucl}(\tau_1-\tau_2) = \bar{G}_{euc}(|\tau_1-\tau_2|) = \theta(\tau_1-\tau_2)\,\bar{G}(\tau_1-\tau_2) + \theta(\tau_2-\tau_1)\bar{G}(\tau_2-\tau_1)$

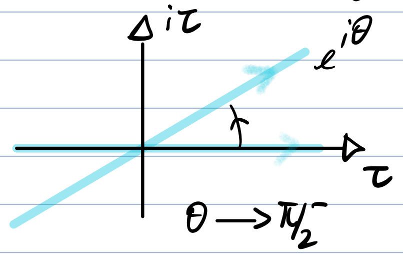
$= \theta(t_1-t_2)\,\bar{G}(\tau_1-\tau_2) + \theta(t_2-t_1)\,\bar{G}(\tau_2-\tau_1)$

$$\boxed{(40)\quad G_{Mink}(t_1-t_2) = \bar{G}(i|t_1-t_2|) \underset{(36)}{=} \frac{e^{-i\Omega|t_1-t_2|}}{2\Omega} = \langle 0|T\,Q(t_1)\,Q(t_2)|0\rangle_{Mink}}$$

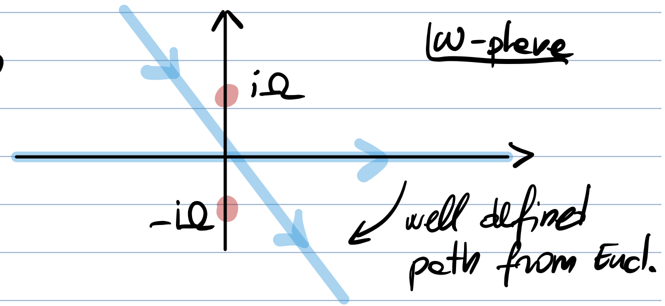
But it is actually instructive to get it also in another way

Let's start from the integral representation of G_{Eucl} in (35) and let's rotate $\tau \to e^{i\theta}\tau$ while rotating anticlockwise $\omega \to \omega e^{-i\theta}$ (so that we are still taking the Fourier transf.)

(41)

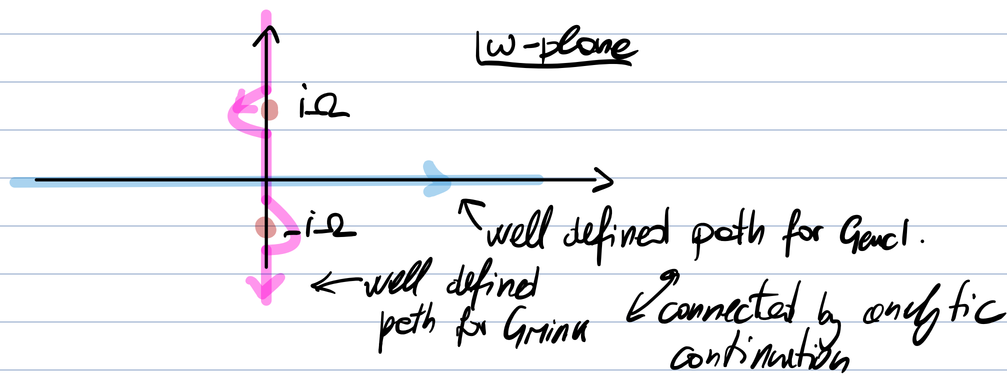


Fourier in ω $\Longrightarrow$



(42) $$\boxed{\tau \longrightarrow it(1-i\varepsilon) \qquad \omega \longrightarrow -i\omega(1+i\varepsilon)}$$

(43)



(44) $$\langle 0|TQ(t_1)Q(t_2)|0\rangle\Big|_{Mink.} = \int \frac{d\omega}{2\pi} \frac{-i\, e^{-i\omega(t_1-t_2)}}{-\omega^2(1+2i\varepsilon)+\Omega^2} = \int d\omega \frac{i\, e^{-i\omega(t_1-t_2)}}{\omega^2-\Omega^2+i\varepsilon}$$

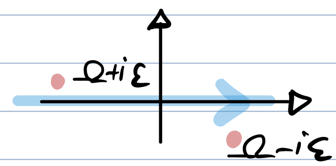
$\omega \to -i\omega(1+i\varepsilon)$

$\tau_1-\tau_2 \to +i(t_1-t_2)(1-i\varepsilon)$

$+$ eq. 35

$2\omega^2\varepsilon \to \varepsilon > 0$

with the usual $\pm i\varepsilon$ analytic structure



so that again one can do via residue Th.

Remark: This is the same result we would obtain starting directly in Mink. while keeping the $i\varepsilon$,

(45) $$S_{Mink} = \int dt\, \frac{1}{2} q(-\partial_t^2-\Omega^2)q + \frac{i\varepsilon}{2}\overbrace{\underbrace{(\dot q^2+\Omega^2 q^2)}_{q(-\partial_t^2+\Omega^2)q}}^{\text{Energy}} = \int dt\, \frac{1}{2} q\overbrace{\left(-\partial_t^2(1+i\varepsilon)-\Omega^2(1-i\varepsilon)\right)}^{D_{Mink}}q$$

$$(46)\quad \begin{cases} Z[J] \propto \int [dq]\, \exp\, i S_{Mink} + \int J\cdot q \propto \exp\left(i J\cdot D_{Mink}^{-1}\cdot J\right) \\ \langle 0|T Q(t_1) Q(t_2)|0\rangle = i\, D_{Mink}^{-1} = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t_1-t_2)} \frac{i}{\omega^2(1+i\varepsilon)-\Omega^2(1-i\varepsilon)} = \int \frac{d\omega}{2\pi}\, e^{-i\omega(t_1-t_2)} \frac{i}{\omega^2-\Omega^2+i\varepsilon} \end{cases}$$

OK

— More Variables (Q_i, P_i): Fields —

So far we have been using only one pair of conjugate variables (Q,P) but nothing prevents us to generalize to more (Q_i, P_j) with

$$(47)\quad [Q_i, Q_j] = 0 \qquad [P_i, P_j] = 0 \qquad [Q_i, P_j] = \delta_{ij}$$

In particular we can work with fields on the same footing (e.g. discretizing space, not only time, and taking the continuum limit in both, which requires adding c.t. ...). Let's take a practical approach and look at a the scalar field

— Scalar Field via Path-integral —

The dynamical variables at a given time slice are the field & it conjugate

$$\boxed{\begin{aligned} &(48)\quad Q_{\vec{x}} = \phi(t,\vec{x}) \qquad P_{\vec{x}} = \pi(t,\vec{x}) \left(= \dot{\phi}(t,\vec{x})\right) = \dot{Q}_{\vec{x}} \\ &(49)\quad \mathcal{L} = \frac{1}{2}(\partial_\mu\phi)^2 - V(\phi) = \frac{1}{2}\dot{\phi}^2 - \frac{(\vec{\nabla}\phi)^2}{2} - V(\phi) \end{aligned}}$$

where the $\vec{x}$ represents the label "i" for each d.o.f., $\boxed{d\phi_{\vec{x}}(t) = d\phi(t,\vec{x}) \quad [D\phi] \propto \prod_{\vec{x}} d\phi_{\vec{x}}(t)}$

$$(50)\quad H = \int d^3x\, \frac{\pi^2(t,\vec{x})}{2} + \frac{(\vec{\nabla}\phi)^2}{2} + V(\phi) = \int d^3x\, \frac{\dot{\phi}^2(t,\vec{x})}{2} + \frac{(\vec{\nabla}\phi)^2}{2} + V(\phi)$$

$$= \int \frac{d^3k}{(2\pi)^3} \frac{1}{2} \hat{\phi}^\dagger(t,\vec{k}) \left[-\partial_t^2 + (m^2 + \vec{k}^2)\right] \hat{\phi}(t,\vec{k}) \quad \leftarrow \text{continuum collection of harmonic oscillators of frequency } \Omega^2 = m^2 + \vec{k}^2$$

$V = \frac{m^2\phi^2}{2} + \text{F.T.}$

$\hookrightarrow V_{|\phi^{n>2}}$ represent anharmonic corrections

Example: free 2pt function

(51) $$\langle 0|T\,\hat\phi(t_1,\vec k)\,\hat\phi(t_2,\vec p)|0\rangle = (2\pi)^3\,\delta^3(\vec k+\vec p)\int e^{-ik^0(t_1-t_2)}\frac{i}{k^{0\,2}-\Omega^2+i\varepsilon}\frac{dk^0}{2\pi}$$

$\Omega^2 \to \vec k^2+m^2$

$\Downarrow$ F.T.

(52) $$\boxed{\langle 0|T\,\phi(t_1,\vec x_1)\,\phi(t_2,\vec x_2)|0\rangle = \int\frac{d^4k}{(2\pi)^4}\,e^{-ik\cdot(x_1-x_2)}\frac{i}{k^2-m^2+i\varepsilon}}$$

Or, more simply perhaps, we have seen that free 2pt function is nothing but the inverse of the diff. K.T. operator D_{MINK} in the Lagrangian, eq. (49), $D^{-1}_{MINK} = \left[\delta^4(x-x')(-\partial_t^2+\vec\nabla^2-m^2+i\varepsilon)\right]^{-1}$ (or in Fourier space $D^{-1}_{MINK} = \left[\delta(t-t')(-\partial_t^2-(\vec k^2+m^2)+i\varepsilon)\right]^{-1}$:

(53) $$\begin{cases}\langle 0|T\,\phi(t_1,\vec x_1)\,\phi(t_2,\vec x_2)|0\rangle = i\,D^{-1}_{MINK} = \int\frac{d^4k}{(2\pi)^4}\,e^{-ik(x_1-x_2)}\frac{i}{k^2-m^2+i\varepsilon}\\ \underbrace{D^{-1}_{MINK}}_{(49)} = \left[\delta(t-t')(-\partial_t^2+\vec\nabla^2-m^2+i\varepsilon)\right]^{-1}\end{cases}$$

More generally & for generic θ, the vacuum-to-vacuum correlator reads

$[D\phi] = \not{\#}\prod_{\vec x} d\phi(t,\vec x)$

(54) $$\boxed{\langle 0|T\,\phi(t_1,\vec x_1)\dots\phi(t_n,\vec x_n)|0\rangle = \frac{\int[D\phi]\exp\{-S[\phi]\}\,\phi(t_1,\vec x_1)\dots\phi(t_n,\vec x_n)}{\int[D\phi]\exp\{-S[\phi]\}}}$$

PATH-INTEGRAL FOR ϕ

Again, with $\theta\to 0$ corresponding to $S_{Eucl.}$ where it is basically stat-mech. and $\theta\to\pi/2^-$ to S_{MINK} obtained thus from $S_{Eucl.}$ with $t\to it(1-i\varepsilon)$

(55) $$\boxed{\langle 0|T\,\phi(x_1)\dots\phi(x_n)|0\rangle_{MINK} = \frac{\int[D\phi]\exp\{i\int d^4x\,(\mathcal{L}+i\varepsilon(\dots))\}\,\phi(x_1)\dots\phi(x_n)}{\int[D\phi]\exp\{i\int d^4x\,(\mathcal{L}+i\varepsilon(\dots))\}}}$$

Remarks:

- The (54)-(55) do not rely on perturbation theory, they are expressed in terms of $S = S_{full}$ (or $\mathcal{L} = \mathcal{L}_{full}$), do not assume $\mathcal{L} = \mathcal{L}_{free} + \mathcal{L}_{int}$. Therefore, they can be at the base of non-perturbative ("lattice") calculations.

- From (54-55), once one has calculated $\langle 0|T\phi ..|0\rangle$, it's possible to extract in principle non-perturbative S-matrix elements via LSZ reduction.

- Sometime one absorbe the normalization $\int[D\phi]\exp\{iS\}$ into $[D\phi]$ so that $\int[D\phi]\exp\{iS\}\cdot 1 = \langle 0|0\rangle = 1$

Functional Methods

Like for the harmonic oscillator, it's useful to introduce the functional generator $Z[J]$ with general S

(56) $$\boxed{Z[J] \equiv \int[D\phi]\,\exp(-S + J\cdot\phi)}$$

shorthand notation for $J\cdot\phi = \int d^4x\, J_i(x)\,\mathcal{O}^i(x)$ or $J_\alpha \mathcal{O}^\alpha$ — both internal & spacetime index

Remark: it's a sort of **Functional Fourier Transform**

(57) $$Z[J] = Z[0]\left(1 + \int d^4x\, J_i(x)\langle \mathcal{O}_i(x)\rangle + \frac{1}{2!}\int d^4x_1 d^4x_2\, J_{i_1}(x_1)\, J_{i_2}(x_2)\langle T\mathcal{O}_{i_1}(x_1)\mathcal{O}_{i_2}(x_2)\rangle + \dots\right.$$

$$= Z[0]\left(\frac{1}{n!} J_{\alpha_1}\dots J_{\alpha_n}\langle T\mathcal{O}_{\alpha_1}\dots\mathcal{O}_{\alpha_n}\rangle\right)$$